Exploring symmetries through the action on the Torsors of the eight connected components Janus Symplectic Group

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Abstract

In this article, after presenting the physical foundations leading to the construction of the Janus Cosmological Model, its principles and consequences, we focus on the Janus symplectic group associated with it. We explore the different symmetries, its action on the various elements of the dual of its Lie algebra, highlighting a charge symmetry, that is, the matter-antimatter duality in both sets of components.

Contents

1 The physico-mathematical foundations of this approach

The French mathematician Jean-Marie Souriau, who passed away in 2012, used to say, *"A little mathematics takes you away from physics, but a lot of it brings you back"*. In his work, he provided an example of such a statement by revealing the physical quantities like energy, momentum, and spin as objects of pure geometry, representing a brilliant application of symplectic geometry. He is one of the few who excelled both as a high-level mathematician and an excellent physicist. In his work *Structure of Dynamical Systems* [35] (today, we prefer to use the term *symplectic groups*), he constructs the action of the Poincaré group on the dual of its Lie algebra, known as the momentum space. It is a vector space of the same dimension as the group, which is 10. He then organizes its components according to:

- A scalar, energy
- A 3-vector momentum
- A 3-vector spin
- A 3-vector to which he gives the name *"passage"*

These components of momentum then define motions in Minkowski space, where the Poincaré group is the isometry group. These motions are divided into classes, and Souriau establishes a connection between particles and classes of motions. He shows that the components of the 3-vector passage can be canceled by choosing a coordinate system that accompanies the particle in its motion. The remaining physical quantities are the first three. Their emergence can also be interpreted as an application of Noether's theorem:

- The scalar energy is then associated with the subgroup of temporal translations.
- The 3-vector momentum with the subgroup of spatial translations.
- The 3-vector spin (unquantized) with the Lorentz subgroup, around which the Poincaré group is constructed.

But at the end of this approach, a surprise awaited the physicist. The *Lorentz group* is defined by:

$$\mathcal{L}or := \{L \in \mathrm{GL}(4,\mathbb{R}),\ \tau(L)L = I_4\}\,.$$

with:

$$\tau(L) := I_{1,3}L^T I_{1,3} \quad , \quad I_{1,k} := \begin{pmatrix} -1 & 0 \\ 0 & I_k \end{pmatrix}\ (k \in \mathbb{N}).$$

We extend the map τ to vectors of $\mathbb{R}^4$, by setting for all $X \in \mathbb{R}^4$:

$$\tau(X) := X^T I_{1,3}$$

The Lorentz group has four connected components (see [7], [27] and [33]):

- $\mathcal{L}or_n$ is the neutral component (its *restricted subgroup*), does not invert either space or time *i.e.* defined by:

$$\mathcal{L}or_n := \{L \in \mathcal{L}or,\ \det(L) = 1\ \wedge\ [L]_{00} \geq 1\}$$

- $\mathcal{L}or_s$ inverts space *i.e.* defined by:

$$\mathcal{L}or_s := \{L \in \mathcal{L}or,\ \det(L) = -1\ \wedge\ [L]_{00} \geq 1\}$$

- $\mathcal{L}or_t$ inverts time but not space *i.e.* defined by:

$$\mathcal{L}or_t := \{L \in \mathcal{L}or,\ \det(L) = 1\ \wedge\ [L]_{00} \leq -1\}$$

- $\mathcal{L}or_{st}$ inverts both space and time *i.e.* defined by:

$$\mathcal{L}or_{st} := \{L \in \mathcal{L}or,\ \det(L) = -1\ \wedge\ [L]_{00} \leq -1\}$$

We have:

$$\mathcal{L}or = \mathcal{L}or_n\ \sqcup\ \mathcal{L}or_s\ \sqcup\ \mathcal{L}or_t\ \sqcup\ \mathcal{L}or_{st}. \tag{1}$$

The first two components are grouped together to form the subgroup called *"orthochronous"*:

$$\mathcal{L}or_o = \mathcal{L}or_n\ \sqcup\ \mathcal{L}or_s \tag{2}$$

It includes **P**-symmetry, which poses no problem for physicists who know that there are photons of *"right"* and *"left"* helicity whose motions are derived from this symmetry. This corresponds to the phenomenon of the polarization of light.

The last two components form the subset *"retrochronous"* or *"antichronous"*, whose components invert time:

$$\mathcal{L}or_a = \mathcal{L}or_t\ \sqcup\ \mathcal{L}or_{st} \tag{3}$$

We have:

$$\mathcal{L}or = \mathcal{L}or_o\ \sqcup\ \mathcal{L}or_a \tag{4}$$

The *Poincaré group* is defined by:

$$\mathcal{P}oin := \left\{ \begin{pmatrix} L & D \\ 0 & 1 \end{pmatrix},\ L \in \mathcal{L}or\ \wedge\ D \in \mathbb{R}^4 \right\}, \tag{5}$$

it inherits the properties of the Lorentz group and thus has four connected components. We then distinguish the subgroup of the complete Poincaré group, constructed from the orthochronous components of the Lorentz group. And we define all components (like Lorentz group):

$$\forall \alpha \in \{n, s, t, st, o, a\},\ \mathcal{P}oin_\alpha := \left\{ \begin{pmatrix} L_\alpha & D \\ 0 & 1 \end{pmatrix},\ L_\alpha \in \mathcal{L}or_\alpha\ \wedge\ D \in \mathbb{R}^4 \right\}. \tag{6}$$

We have the same decomposition like (1), (2), (3) and (4).

The classification of motions yields two classes corresponding to the movements of photons and particles with a positive mass m. Souriau summarizes his study by providing a summary of the group's action on its momentum (see [35] chapter 13).

We can define the moment matrix M and the stress–energy vector P as follows:

$$M := \begin{pmatrix} 0 & g^T \\ g & j(\ell) \end{pmatrix} \quad , \quad P := \begin{pmatrix} E \\ p \end{pmatrix}$$

with ℓ the angular momentum of M, g the relativist barycenter of M, p the linear momentum of P, and E the energy of P.

The action is written (see [35] equation 13.107) for all $L \in \mathcal{L}or$:

$$M' = LM\tau(L) + C\tau(P)\tau(L) - CLP\tau(C) \tag{7}$$

$$P' = LP \tag{8}$$

We have:

$$\mathcal{L}or_t = -\mathcal{L}or_s \quad \mathcal{L}or_{st} = -\mathcal{L}or_n \tag{9}$$

Then, it is possible to write the complete Poincaré group as:

$$\mathcal{P}oin := \left\{ \begin{pmatrix} \lambda L_o & D \\ 0 & 1 \end{pmatrix}, \ L_o \in \mathcal{L}or_o \ \wedge \ D \in \mathbb{R}^4 \ \wedge \ \lambda \in \{\pm 1\} \right\}. \tag{10}$$

The action of the complete group is then written as follows for all $L := \lambda L_o \in \mathcal{L}or$:

$$M' = L_o M\tau(L_o) + \lambda C\tau(P)\tau(L_o) - CLP\tau(C)$$

$$P' = \lambda L_o P$$

It's then observed that the *retrochronous* components reverse the energy and, consequently, the mass, as noted by J.M. Souriau ((14.67) of page 198 [35]).

In the past, we have seen an example where P. Dirac suggested the use of an electric charge symmetry. The existence of particles with opposite electric charges was thus directly implied by an extension of the theory. This involved postulating the existence of positrons. Fortunately, the existence of such particles was quickly confirmed by C.A. Anderson's observations[1], which earned him the Nobel Prize in 1936.

We are in 1970. J.M. Souriau's theoretical framework raised the possibility of particles with negative energy, which were categorized into two classes:

- Particles endowed with a negative mass m
- Photons endowed with negative energy.

In conclusion, the author indicated potential measures to circumvent the emergence of particles with negative mass, one of which was to decide that only the orthochronous components of the Poincaré group should pertain to the realm of physics.

[1] To be precise, this observation did not follow P. Dirac's deduction in the sense that, in 1923, the Russian D. Skobeltzyn was the first to make this observation.

2 When observations suggest the introduction of negative masses

Major Implications in Cosmology

When observations suggest the introduction of negative masses in cosmology, the introduction of particles with negative mass and energy posed a serious problem in physics. It is noteworthy that they also appear in quantum mechanics if, in the field theory of quantum mechanics, we choose to make the **T**-operator, which inverts time, linear and unitary, instead of antiunitary and antilinear as had been chosen until then, precisely to oppose the emergence of states with negative energy [41]. This perspective opens up a new field of theoretical research where N. Debergh has published some articles [18], [17]. However, it required an observation to reveal a phenomenon that demonstrates the action of unknown particles. The discovery of the acceleration of cosmic expansion, due to negative contents, can be considered an answer to this question.

In general relativity, the cosmological dynamics are determined by two possible causes: the value of the cosmological constant present in Einstein's equation, and a field source represented by the contents of the tensor on the right-hand side of the equation. In mixed notation:

$$R^{\nu}_{\mu} - \frac{1}{2} R \delta^{\nu}_{\mu} = \chi T^{\nu}_{\mu} - \chi \Lambda \delta^{\nu}_{\mu} \tag{11}$$

In this equation, R^{ν}_{μ} represents the Ricci curvature tensor, R is the Ricci scalar, and δ^{ν}_{μ} is the Kronecker symbol, which is used to write the equations in a compact form. The term χT^{ν}_{μ} corresponds to the energy-momentum tensor that describes the distribution and flow of energy and momentum in spacetime, which acts as the source of the gravitational field. The term $-\chi\Lambda\delta^{\nu}_{\mu}$ introduces the cosmological constant Λ, which can be interpreted as the energy density of the vacuum of space contributing to the overall dynamics of the cosmos.

Taking $\Lambda = 0$, assuming positive masses, and by considering the cosmic fluid as a perfect gas, the source tensor, or the energy-momentum tensor, is given by:

$$T^{\nu}_{\mu} = \begin{pmatrix} \rho c^2 & 0 & 0 & 0 \\ 0 & -p & 0 & 0 \\ 0 & 0 & -p & 0 \\ 0 & 0 & 0 & -p \end{pmatrix} \tag{12}$$

where c represents the speed of light, ρ is the energy density and p is the pressure, as the volumetric kinetic energy density associated with thermal agitation at the mean quadratic velocity $\langle v \rangle$, following the relationship:

$$p = \frac{\rho \langle v^2 \rangle}{3} \tag{13}$$

If it involves positive mass, then ρ and p are positive. However, the evolution then indicates deceleration. The currently adopted solution attributes this expansion's acceleration to the constant Λ, likened to the action of dark energy, of unknown nature, with constant volumetric density. The evolution of the cosmic space scale factor then becomes exponential.

Deceleration could occur if the source of the field could be attributed to a negative mass and energy content, while simultaneously taking a zero cosmological constant. This results in an evolution with a negative curvature index that tends, at infinity, towards asymptotic expansion at constant speed. However, a significant and insurmountable difficulty arises. With a single field equation, test particles, especially test masses, follow the same geodesics. This can be summarized in the laws of interaction:

- Positive masses (source ρ and $p > 0$) attract both positive and negative masses.
- Negative masses (source ρ and $p < 0$) repel both positive and negative masses.

Such a set of forces violates both the action-reaction principle and the equivalence principle, giving rise to the uncontrollable *runaway phenomenon*. When this aspect was discovered in 1957 [5], scientists concluded that negative masses could never be a part of physics.

J.P. Petit and G. D'Agostini ([30], [29]) consider a bimetric configuration, where positive masses and positive-energy photons would follow geodesics arising from a metric $g_{\mu\nu}^{(+)}$, while negative masses and negative-energy photons would follow geodesics arising from a metric $g_{\mu\nu}^{(-)}$. This led to the project of conceiving a system of two coupled field equations derived from an action. However, the first attempt at a bimetric modeling was by T. Damour and I. Kogan in 2004 ([12], [13]). Starting from these two metrics, two Ricci tensors $R_{\mu\nu}^{(+)}$ and $R_{\mu\nu}^{(-)}$, and two Ricci scalars $R^{(+)}$ and $R^{(-)}$ are constructed. The proposed system, with the pair of metrics $g_{\mu\nu}^{(L)}$ and $g_{\mu\nu}^{(R)}$, is then:

$$R_{\mu\nu}^{L} - \frac{1}{2} g_{\mu\nu}^{L} R^{L} = \left(T_{\mu\nu}^{L} + t_{\mu\nu}^{L}\right) \tag{14}$$

$$R_{\mu\nu}^{R} - \frac{1}{2} g_{\mu\nu}^{R} R^{R} = \left(T_{\mu\nu}^{R} + t_{\mu\nu}^{R}\right) \tag{15}$$

Such an approach implies introducing a different geometric context. It does not involve introducing negative masses at all. The authors consider an interaction between two branes, using gravitons with a mass spectrum. The tensors T^L and T^R represent the content of each brane. The tensors t^L and t^R can be described as interaction tensors. Tensor t^L reflects the source of the action of particles from the "Right" brane structured by equation 15 on those from the "Left" brane structured by the first field equation 14, while tensor t^R represents the source of the action of particles from the "Left" brane structured by the first field equation 14 on those from the "Right" brane structured by equation 15.

Starting from the metrics $g_{\mu\nu}^{(L)}$ and $g_{\mu\nu}^{(R)}$, we construct covariant derivative operators ∇_L^ν and ∇_R^ν. The covariant derivatives of the first two terms are zero by construction. Therefore, the covariant derivatives of the second terms must also be zero. This represents the translation of the *Bianchi conditions*.

It is easy to envision, with mixed notation:

$$T^{L\,\nu}_{\ \ \mu} = \begin{pmatrix} \rho_L c^2 & 0 & 0 & 0 \\ 0 & -p_L & 0 & 0 \\ 0 & 0 & -p_L & 0 \\ 0 & 0 & 0 & -p_L \end{pmatrix}, \quad T^{R\,\nu}_{\ \ \mu} = \begin{pmatrix} \rho_R c^2 & 0 & 0 & 0 \\ 0 & -p_R & 0 & 0 \\ 0 & 0 & -p_R & 0 \\ 0 & 0 & 0 & -p_R \end{pmatrix} \tag{16}$$

And to have:

$$\nabla^\nu_L T^{L\,\nu}_{\ \ \mu} = \nabla^\nu_R T^{R\,\nu}_{\ \ \mu} = 0 \tag{17}$$

which are then conservation equations. The problem then is to produce the forms of the interaction tensors such as:

$$\nabla^\nu_L t^{L\,\nu}_{\ \ \mu} = \nabla^\nu_R t^{R\,\nu}_{\ \ \mu} = 0 \tag{18}$$

Here ends the attempt of these two authors.

A more intriguing approach in 2008 is that of S. Hossenfelder [21], also based on an action, which leads to the system of equations:

$$^{(g)}R_{vk} - \frac{1}{2} g_{vk}\, {}^{(g)}R = T_{kv} - \underline{V} \sqrt{\frac{h}{g}}\, a^{\underline{v}}_{v} a^{\underline{k}}_{k} \tag{19}$$

$$^{(\underline{h})}R_{\underline{vk}} - \frac{1}{2} h_{\underline{vk}}\, {}^{(\underline{h})}R = \underline{T}_{\underline{vk}} - W \sqrt{\frac{g}{\underline{h}}}\, a^{k}_{\underline{k}} a^{v}_{\underline{v}} T_{kv} \tag{20}$$

The same divergence-free first terms are found, whose form results from the introduction of the Ricci scalars $^{(\underline{h})}R$ and $^{(g)}R$ into the action that was used to construct this system of equations. The second terms of both right-hand sides also reveal interaction tensors, weighted by the square root of the ratio of the determinants of the two metrics, whose presence stems from the form of the considered action. The author indicates his intention to manage masses of both signs. However, his system of equations generates interaction laws, similar to what was obtained with Einstein's equation, which are also physically unacceptable. Believing that this aspect is automatically linked to any bimetric description, the author then abandons this project.

It is useful, in presenting a new model, to follow the progression adopted by the authors of what would become the Janus cosmological model. While the first two attempts were immediately based on an action, their approach can be described as heuristic.

In 1995 [28], J.P. Petit created numerical simulations by making positive and negative masses interact. Starting from complete symmetry, where only the sign of the mass changes, he observed a percolation phenomenon that hardly seems comparable to observational data as on figure 1.

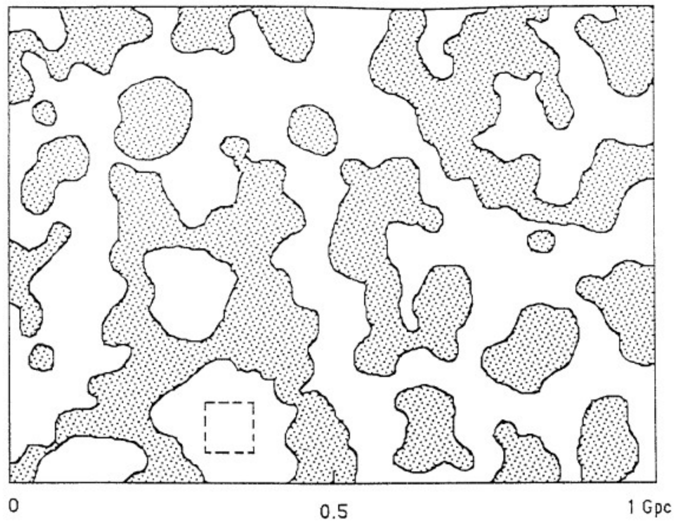

Figure 1: Percolation phenomenon with separation of masses of opposite signs

He then had the idea to endow the negative species with a higher volumetric mass value. A lacunar structure then appears, where the positive mass is caught between conglomerates of adjacent negative mass as on figure 2.

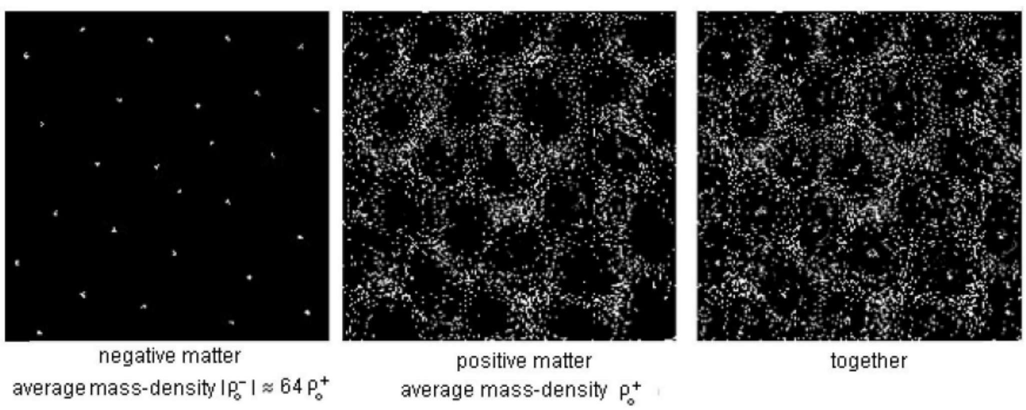

Figure 2: Formation of a Lacunar Structure

Today, it is known that ordinary, visible matter has such a lacunar structure. In 1995, the author concluded that this structure results from a profound asymmetry between positive and negative masses. Since the accretion time, or the time it takes for structures to form through gravitational instability, varies as the inverse of the square root of the absolute value of the volumetric mass, it is deduced that conglomerates of negative mass are the first to form, in a regular distribution, confining positive mass within the interstitial space.

The existence of these conglomerates of negative mass was confirmed with the discovery in 2017 of the first among them, the dipole repeller 3 [20].

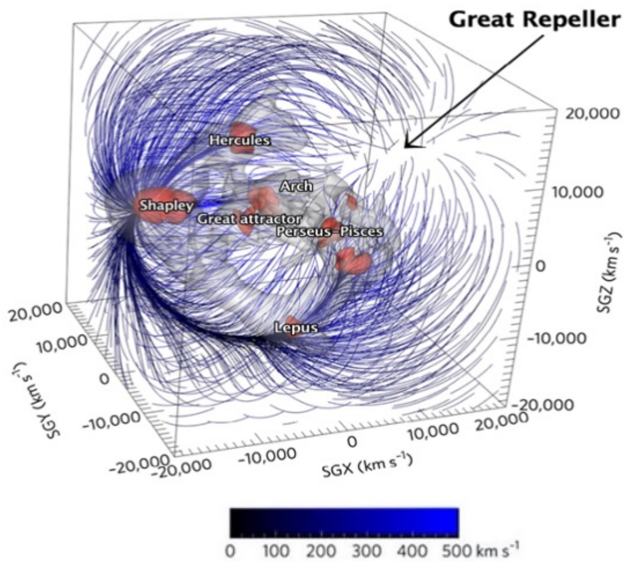

Figure 3: Dipole Repeller

The second point in favor of such a model is its prediction of an early formation of stars and galaxies, within the first hundred million years, based on the following idea: when positive mass adopts its lacunar structure, it is violently compressed by the repulsion exerted by the two adjacent negative mass conglomerates. It then heats up, but its flat plate structure leads to equally abrupt cooling by radiative dissipation, which destabilizes it and results in the birth of stars and galaxies (figure 4).

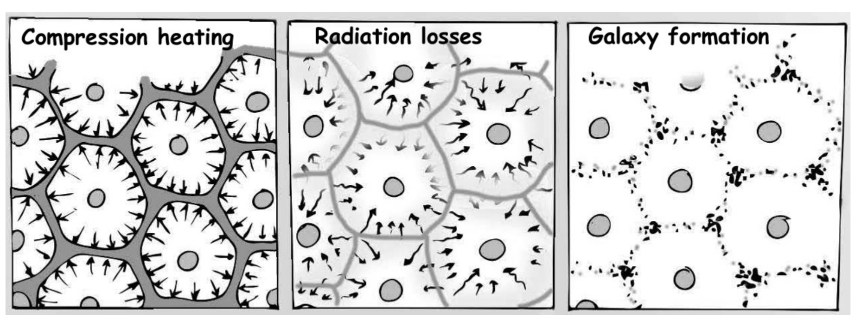

Figure 4: Scheme for the rapid formation of galaxies

Observational confirmation has recently been provided in 2022 [19] and 2023 [1], with the James Webb Space Telescope revealing the existence of fully formed spiral galaxies, hosting stars that are already old, only 500 million years post Big Bang, a scenario that the prevailing model, even with the best parameters for dark matter, cannot account for.

Over the years, observational confirmations have become increasingly numerous, prompting the development of a mathematically coherent relativistic model. The first success was achieved in 2014 [31], by constructing a coupled field equation system that accounts for the acceleration of cosmic expansion, under the assumption that both the positive and negative sectors satisfy the conditions of homogeneity and isotropy. Under these conditions, the corresponding hyperbolic Riemannian metrics are FLRW-type metrics:

$$g^{(+)}_{\mu\nu} = dx^{0^2} - \frac{a^{(+)^2}}{1-k^{(+)}}\left(du^2 + u^2 d\theta^2 + u^2 \sin^2\theta d\phi^2\right) \tag{21}$$

$$g^{(-)}_{\mu\nu} = dx^{0^2} - \frac{a^{(-)^2}}{1-k^{(-)}}\left(du^2 + u^2 d\theta^2 + u^2 \sin^2\theta d\phi^2\right) \tag{22}$$

The set $\{u, \theta, \phi\}$ represents the radial coordinates. x^0 is a chronological variable. $a^{(+)}$ and $a^{(-)}$ are the scale factors for the two populations. A system of equations of the form:

$$R^{(+)\nu}_{\ \ \mu} - \frac{1}{2}R^{(+)}\delta^{\nu}_{\mu} = \chi\left[T^{(+)\nu}_{\ \ \mu} + T^{(-)\nu}_{\ \ \mu}\right] \tag{23}$$

$$R^{(-)\nu}_{\ \ \mu} - \frac{1}{2}R^{(-)}\delta^{\nu}_{\mu} = -\chi\left[T^{(-)\nu}_{\ \ \mu} + T^{(+)\nu}_{\ \ \mu}\right] \tag{24}$$

Applied specifically to this symmetry, this system generates a solution such that $a^{(+)} = a^{(-)}$ with equal and opposite specific masses, which cannot account for the results of the simulations. The authors then attempt to enrich their model by writing:

$$R^{(+)\nu}_{\ \ \mu} - \frac{1}{2}R^{(+)}\delta^{\nu}_{\mu} = \chi\left[T^{(+)\nu}_{\ \ \mu} + \Phi(x^0)T^{(-)\nu}_{\ \ \mu}\right] \tag{25}$$

$$R^{(-)\nu}_{\ \ \mu} - \frac{1}{2}R^{(-)}\delta^{\nu}_{\mu} = -\chi\left[T^{(-)\nu}_{\ \ \mu} + \phi(x^0)T^{(+)\nu}_{\ \ \mu}\right] \tag{26}$$

In these equations, all terms depend only on the chronological variable x^0. With this approach, where only solutions $a^{(+)}$ and $a^{(-)}$ are sought as functions of the chronological variable x^0, the covariant derivatives refer only to this single variable. The consistency of the equations leads to the result:

$$\Phi = \left[\frac{a^{(-)}}{a^{(+)}}\right]^3 = \phi^{-1} \tag{27}$$

As the determinants of the metrics are:

$$g^{(+)} = -a^{(+)^6}\sin^2\theta \tag{28}$$

$$g^{(-)} = -a^{(-)^6}\sin^2\theta \tag{29}$$

It is then noticed that the coefficients ϕ and Φ are respectively identified with the factor $\sqrt{\frac{g^{(+)}}{g^{(-)}}}$ and its inverse, which draws attention to the essay by D. Hossenfelder [12]. In 2014, an exact solution is presented [31], and in 2018, G. D'Agostini [11] shows that it indeed accounts for the available observational data (5).

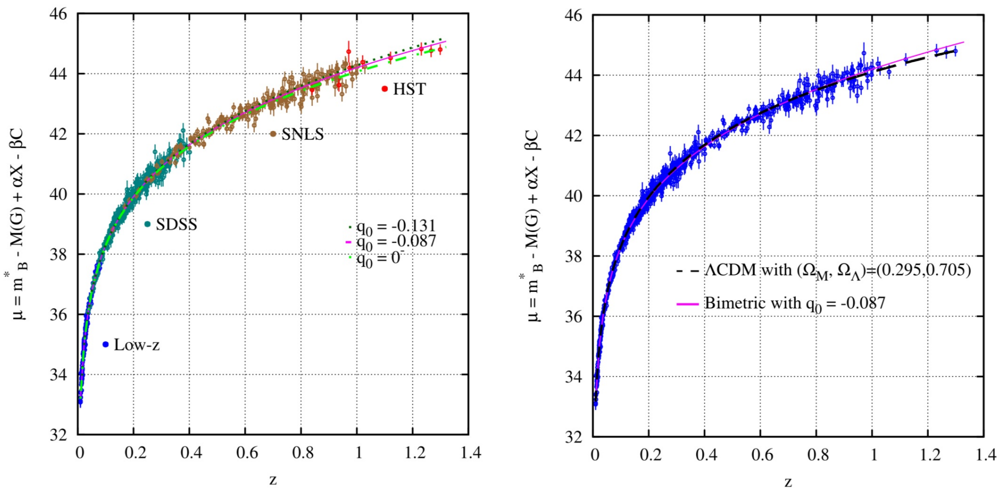

Figure 5: Acceleration of the expansion in agreement with observational data

Thus, the system of coupled field equations for the Janus model becomes:

$$R^{(+)}_{\mu\nu} - \frac{1}{2} R^{(+)} g^{(+)}_{\mu\nu} = \chi \left[T^{(+)}_{\mu\nu} + \sqrt{\frac{g^{(-)}}{g^{(+)}}} \hat{T}^{(-)}_{\mu\nu} \right] \tag{30}$$

$$R^{(-)}_{\mu\nu} - \frac{1}{2} R^{(-)} g^{(-)}_{\mu\nu} = \kappa\chi \left[\sqrt{\frac{g^{(+)}}{g^{(-)}}} \hat{T}^{(+)}_{\mu\nu} + T^{(-)}_{\mu\nu} \right] \tag{31}$$

Major Implications of Quantum Field Theory

In Quantum Mechanics, quantum physicists traditionally adopt an anti-linear and anti-unitary perspective for the **T**-operator. A **P**-operator is chosen to be unitary and linear for analogous reasons as quoted on pages 75 and 76 of [41]:

> If **P** were antiunitary then it would anticommute with i, so $\mathbf{P}H\mathbf{P}^{-1} = -H$. But then for any state Ψ of energy $E \geq 0$, there would have to be another state $\mathbf{P}^{-1}\Psi$ of energy $-E < 0$. **There are no states of negative energy (energy less than that of the vacuum), so we are forced to choose the other alternative: P is linear and unitary, and commutes rather than anticommutes with H.** On the other hand, setting $\rho = 0$ in Eq. (2.6.6) yields
>
> $$\mathbf{T} iH \mathbf{T}^{-1} = -iH.$$
>
> If we supposed that **T** is linear and unitary then we could simply cancel the is, and find $\mathbf{T}H\mathbf{T}^{-1} = -H$, **with the again disastrous conclusion that for any state Ψ of energy E there is another state $\mathbf{T}^{-1}\Psi$ of energy $-E$. To avoid this, we are forced here to conclude that T is antiunitary and antilinear.**

> Now that we have decided that **P** is linear and **T** is antilinear, we can conveniently rewrite Eqs. (2.6.3)-(2.6.6) in terms of the generators (2.4.15)-(2.4.17) in a three-dimensional notation.

And to conclude by adding on page 104 that: **"No examples are known of particles that furnish unconventional representations of inversions, so these possibilities will not be pursued further here. From now on, the inversions will be assumed to have the conventional action assumed in Section 2.6"**. So, at the time this theory was conceived, no phenomena had yet been observed that would allow us to address the question of negative energy states. However, the observation that the universe's expansion is accelerating compels us to consider these states, as pointed out on the previous section and in the study [31]. This idea was subsequently taken up and developped by N. Debergh ([18], [17]).

In summary, everything mentioned above is intended to recall the approach used to create this model, which presents itself as an extension of general relativity, exploring its physical foundations, implications, and outcomes. It is now appropriate to attempt to specify the contents of this model. A presentation of this symplectic group, which was associated with it, has already been published ([31]), showing that matter-antimatter symmetry could be included. The second paper, which will follow the publication of this present article revised in accordance with your feedback, is currently being drafted in a more structured manner. It will include the Lagrangian derivation from an action, as illustrated in the diagram proposed in reference [32].

3 When the theory of dynamic groups illuminates the traveled path

The application of the coadjoint action of a symplectic group on the dual of its Lie algebra, initiated by the mathematician Jean-Marie Souriau, has shed light on specific aspects of the approach followed by physics. The orbit method is due to Kirillov ([6], [10], [8], [9], [23], [24], [26], [36], [39] and [40]).

Thus, the restricted Lorentz symplectic group, limited to its two orthochrone components, translates, through the invariance properties that result from it, the aspects of special relativity. In 1970, J-M Souriau established that the analysis of the components of its moment makes it possible to shed light on the geometric nature of a spin (not quantized): see [35] and [34]. He uses for this purpose symplectic methods ([16], [14], [37] and [38]). In the theory of symplectic groups, we show a classification in terms of movements. At this stage, the action of these elements reversing space finds its illustration in the phenomenon of polarization of light, where any "right" photon can be converted into a "left" photon.

By operating the product of this group by that of the spatio-temporal translations, we obtain the restricted Poincaré symplectic group, always limited to its two orthochrone components. In its moment, we first find the energy related to the subgroup of temporal translations. Then the momentum, linked to the spatial translations, the two being linked by the invariance of the modulus of the energy-momentum four-vector under the action of the Lorentz group.

By adding a translation along a fifth dimension to the restricted Poincaré group, we form a Lie group to which we will give the name *Restricted Kaluza Group* ([2], [3], [4], [22], [25]). This group is not the 15-dimensional Kaluza group associated with a 5-dimensional Lorentzian manifold but a new 11-dimensional group, including 5-dimensional space-time translation. This new dimension endows the momentum with an additional scalar that can be identified with the electric charge q, which may be positive, negative, or zero, and is still not quantized. We then bring out the geometric translation according to a scalar ϕ due to endowing the masses with an invariant electric charge. Then, by bringing in a new symmetry reflecting the inversion of the fifth dimension, synonymous with an inversion of the scalar from q to $-q$, we double the number of its connected components from 2 to 4. The action on the moment then links this new symmetry to the inversion of the electric charge q. We thus deduce the geometric modeling of charge conjugation or **C**-*Symmetry*, which translates the matter-antimatter symmetry introduced by Dirac. It's then logical to name this new extension, the *Restricted Janus Group*.

By introducing a new symmetry to the previous group, which we describe as **T**-*Symmetry* and which converts matter into antimatter with negative mass – a concept we could name *antimatter in the Feynman sense* – we build the *Janus Symplectic Group*. Thus, we double the number of connected components from four to eight, grouped into two subsets: *"Orthochronous"*, conserving time and energy properties, and *"Antichronous"*, reversing time and energy. Therefore, we bring forth the geometric translation of endowing masses with an invariant electric charge. As the Jean-Marie Souriau demonstrated as early as 1970, a pioneer in the theory of symplectic groups ([35], [15], [34]), this approach has allowed key elements, which have marked the progress of relativistic physics, to be given a purely geometric nature.

In relation to the world of physics, wouldn't the role of mathematics be to illuminate the path traveled? Conversely, could it be possible that the exploration of new symmetries, accompanying this decoding using symplectic groups, contains more than what we thought we put into it? That it could designate new paths to follow?

This is what we will consider with the Janus Symplectic Group with charge symmetry, by integrating the antichronous components of the Lorentz group, resulting from its simple axiomatic definition, with the obvious repercussions on the Poincaré group and its extensions.

4 Janus Symplectic Group

Let $\tilde{\mathbf{T}} := I_{1,3}$, $\tilde{\mathbf{P}} := -\tilde{\mathbf{T}}$ and:

$$\forall \lambda, \nu \in \{0,1\}, \quad \mathcal{L}or\left(\tilde{\mathbf{P}}^{\nu}\tilde{\mathbf{T}}^{\lambda}\right) := \left\{L_n \tilde{\mathbf{P}}^{\nu}\tilde{\mathbf{T}}^{\lambda},\ L_n \in \mathcal{L}or_n\right\}.$$

Then, there are 4 connected components of $\mathcal{L}or$, given by[2]

$$\mathcal{L}or_n = \mathcal{L}or\left(\tilde{\mathbf{P}}^0\tilde{\mathbf{T}}^0\right) \qquad \mathcal{L}or_s = \mathcal{L}or\left(\tilde{\mathbf{P}}^1\tilde{\mathbf{T}}^0\right)$$
$$\mathcal{L}or_t = \mathcal{L}or\left(\tilde{\mathbf{P}}^0\tilde{\mathbf{T}}^1\right) \qquad \mathcal{L}or_{st} = \mathcal{L}or\left(\tilde{\mathbf{P}}^1\tilde{\mathbf{T}}^1\right)$$

and we have the decomposition:

$$\mathcal{L}or = \bigsqcup_{\nu,\lambda\in\{0,1\}} \mathcal{L}or\left(\tilde{\mathbf{P}}^\nu\tilde{\mathbf{T}}^\lambda\right) \tag{32}$$

Then, we define the Janus symplectic group.

Definition 4.1. The *Janus symplectic group* is defined as the subgroup of $\mathrm{GL}(6,\mathbb{R})$:

$$\mathcal{J}an := \left\{\begin{pmatrix} L & 0 & D \\ 0 & (-1)^\eta & \phi \\ 0 & 0 & 1 \end{pmatrix},\ \eta\in\{0,1\}\ \wedge\ \phi\in\mathbb{R}\ \wedge\ L\in\mathcal{L}or\ \wedge\ D\in\mathbb{R}^4\right\}$$

The Janus symplectic group is therefore a subgroup of the group of isometries in dimension 5 given by[3]:

$$\mathrm{Aff}(\mathcal{O}(1,4)) := \left\{\begin{pmatrix} L & D' \\ 0 & 1 \end{pmatrix},\ L\in\mathcal{O}(1,4)\ \wedge\ D'\in\mathbb{R}^5\right\}$$

with $\tau_{1,4}(L) := I_{1,4}L^T I_{1,4}$ and $\mathcal{O}(1,4) := \{L\in\mathrm{GL}(5,\mathbb{R}),\ \tau_{1,4}(L)L = I_5\}$. The elements of $\mathrm{Aff}(\mathcal{O}(1,4))$ are the elements which preserve the distance between two events (*pentavectors*) $X := (t,x,y,z,\xi)$ and $X' := (t',x',y',z',\xi')$ given by:

$$d(X,X') := c^2(t-t')^2 - (x-x')^2 - (y-y')^2 - (z-z')^2 - (\xi-\xi')^2$$

This fifth dimension is of space type (we note the variable ξ). Each dimension is therefore associated with a symmetry, there are three types of symetries:

- the $\mathbf{T}$-*symmetry*;
- the $\mathbf{P}_x$-*symmetry*, $\mathbf{P}_y$-*symmetry*, $\mathbf{P}_z$-*symmetry* grouped together what we call the $\mathbf{P}$-*symmetry*;
- the ξ-*symmetry* corresponding to the $\mathbf{C}$-*symmetry* (the charge conjugation).

[2] Equalities are shown by double inclusion. For example, let's demonstrate that $\mathcal{L}or_s = \mathcal{L}or\left(\tilde{\mathbf{P}}^1\tilde{\mathbf{T}}^0\right)$. Take $L\in\mathcal{L}or_s$ ($\det(L) = -1$ et $[L]_{00}\geq 1$). Then we have $\det(L\tilde{\mathbf{P}}) = -1$ and $[L\tilde{\mathbf{P}}]_{00}\geq 1$ *i.e.*, we have $L_n := L\tilde{\mathbf{P}}\in\mathcal{L}or_n$. Since $\tilde{\mathbf{P}}^{-1} = \tilde{\mathbf{P}}$, we can conclude that $L = L_n\tilde{\mathbf{P}}\in\mathcal{L}or\left(\tilde{\mathbf{P}}^1\tilde{\mathbf{T}}^0\right)$. The inclusion in the other direction is trivial.

[3] $\mathrm{Aff}(\mathcal{O}(1,4))$ is the affine group associated with $\mathcal{O}(1,4)$, it is also defined by the semi-direct product $\mathrm{Aff}(\mathcal{O}(1,4)) := \mathcal{O}(1,4)\ltimes\mathbb{R}^5$. We can also define the symplectic Janus group as being the affine group associated with the subgroup of $\mathcal{O}(1,4)$ given by:

$$\mathcal{E}lec := \left\{\begin{pmatrix} L & 0 \\ 0 & (-1)^\eta \end{pmatrix},\ \eta\in\{0,1\}\ \wedge\ L\in\mathcal{L}or\right\}$$

called the *symplectic electric group* and we have $\mathcal{J}an := \mathrm{Aff}(\mathcal{E}lec)$.

This space of dimension 5 is a Minkowski metric space to which we have added one dimension, it has the metric $I_{1,4}$.

We also define the *restricted Janus group* is the subgroup of $\mathcal{J}an$ given by:

$$\mathcal{J}an_n := \left\{ \begin{pmatrix} L_n & 0 & D \\ 0 & 1 & \phi \\ 0 & 0 & 1 \end{pmatrix},\ \phi \in \mathbb{R} \ \wedge \ L_n \in \mathcal{L}or_n \ \wedge \ D \in \mathbb{R}^4 \right\}$$

Let:

$$\mathbf{C} := \begin{pmatrix} I_4 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad , \quad \mathbf{P} := \begin{pmatrix} \tilde{\mathbf{P}} & 0 \\ 0 & I_2 \end{pmatrix} \quad , \quad \mathbf{T} := \begin{pmatrix} \tilde{\mathbf{T}} & 0 \\ 0 & I_2 \end{pmatrix}.$$

We have:

$$\forall \lambda, \eta, \nu \in \{0,1\},\ \begin{pmatrix} L_n & 0 & D \\ 0 & 1 & \phi \\ 0 & 0 & 1 \end{pmatrix} \mathbf{C}^\eta \mathbf{P}^\nu \mathbf{T}^\lambda = \begin{pmatrix} L_n \tilde{\mathbf{P}}^\nu \tilde{\mathbf{T}}^\lambda & 0 & D \\ 0 & (-1)^\eta & \phi \\ 0 & 0 & 1 \end{pmatrix}$$

and therefore by equation (32):

$$\mathcal{J}an = \left\{ \begin{pmatrix} L_n \tilde{\mathbf{P}}^\nu \tilde{\mathbf{T}}^\lambda & 0 & D \\ 0 & (-1)^\eta & \phi \\ 0 & 0 & 1 \end{pmatrix},\ \lambda, \eta, \nu \in \{0,1\} \ \wedge \ \phi \in \mathbb{R} \ \wedge \ L_n \in \mathcal{L}or_n \ \wedge \ D \in \mathbb{R}^4 \right\}.$$

Definition 4.2. (i) The ***CPT**-group* is the subgroup $\mathcal{K}$ of $\mathcal{J}an$ of order 8 generated by $\mathbf{C}$, $\mathbf{P}$ and $\mathbf{T}$ *ie*:

$$\mathcal{K} := \left\{ \mathbf{C}^\eta \mathbf{P}^\nu \mathbf{T}^\lambda,\ \eta, \nu, \lambda \in \{0,1\} \right\} = \{I_6, \mathbf{T}, \mathbf{P}, \mathbf{PT}, \mathbf{C}, \mathbf{CT}, \mathbf{CP}, \mathbf{CPT}\}.$$

(ii) For all $\mathbf{X} \in \mathcal{K}$, the ***X**-component of $\mathcal{J}an$* is:

$$\mathcal{J}an\,(\mathbf{X}) := \{JX,\ J \in \mathcal{J}an_n\}.$$

Thus, we have:

$$\mathcal{J}an\left(\mathbf{C}^\eta \mathbf{P}^\nu \mathbf{T}^\lambda\right) = \left\{ \begin{pmatrix} L_n \tilde{\mathbf{P}}^\nu \tilde{\mathbf{T}}^\lambda & 0 & D \\ 0 & (-1)^\eta & \phi \\ 0 & 0 & 1 \end{pmatrix},\ \phi \in \mathbb{R} \ \wedge \ L_n \in \mathcal{L}or_n \ \wedge \ D \in \mathbb{R}^4 \right\}.$$

These 8 components are the 8 connected components of $\mathcal{J}an$, we have the decomposition:

$$\mathcal{J}an = \bigsqcup_{\mathbf{X} \in \mathcal{K}} \mathcal{J}an\,(\mathbf{X}) = \bigsqcup_{\eta, \nu, \lambda \in \{0,1\}} \mathcal{J}an\left(\mathbf{C}^\eta \mathbf{P}^\nu \mathbf{T}^\lambda\right).$$

The group $\mathcal{L}or$ is the Lie group of dimension 6 and its Lie algebra is:

$$\mathfrak{lor} := \mathcal{A}(1,3) := \{\Lambda \in \mathcal{M}(4, \mathbb{R}),\ \tau_{1,3}(\Lambda) = -\Lambda\}$$

Then, the group $\mathcal{J}an$ is a Lie group of dimension 11 and its Lie algebra is:

$$\mathfrak{jan} = \left\{ \begin{pmatrix} \Lambda & 0 & \Gamma \\ 0 & 0 & \varepsilon \\ 0 & 0 & 0 \end{pmatrix},\ \Lambda \in \mathcal{A}(1,3)\ \wedge\ \Gamma \in \mathbb{R}^4\ \wedge\ \varepsilon \in \mathbb{R} \right\}.$$

We have two caracterisations[4]:

$$\begin{aligned} \left(\mathbb{R}^5\right)^* &= \left\{ \begin{pmatrix} \Gamma \\ \varepsilon \end{pmatrix} \longmapsto - \begin{pmatrix} P^T & q \end{pmatrix} I_{1,4} \begin{pmatrix} \Gamma \\ \varepsilon \end{pmatrix} = -\tau(P)\Gamma - q\varepsilon,\ \begin{pmatrix} P \\ q \end{pmatrix} \in \mathbb{R}^5 \right\} \\ \mathcal{A}(1,3)^* &= \left\{ \Lambda \longmapsto -\frac{1}{2}\mathrm{Tr}(M\Lambda),\ M \in \mathcal{A}(1,3) \right\} \end{aligned}$$

Then, we have:

$$\mathfrak{jan}^* = \left\{ \left\{ \ M \ \middle|\ P \ \middle|\ q \ \right\} : \begin{pmatrix} \Lambda & 0 & \Gamma \\ 0 & 0 & \varepsilon \\ 0 & 0 & 0 \end{pmatrix} \longmapsto -\frac{1}{2}\mathrm{Tr}(M\Lambda) - \tau(P)\Gamma - q\varepsilon,\ M \in \mathcal{A}(1,3)\ \wedge\ P \in \mathbb{R}^4\ \wedge\ q \in \mathbb{R} \right\}^{5}.$$

The *action of the group $\mathcal{J}an$ on* $\mathfrak{jan}^*$ is defined by the coadjoint representation *i.e.*, for any $a \in \mathcal{J}an$ and any $\mu \in \mathfrak{jan}^*$, we denote this action by:

$$a \bullet \mu := \mathrm{Ad}^*_a(\mu).$$

with

$$\begin{array}{cccc} \mathrm{Ad}^* : & \mathcal{J}an & \longrightarrow & \mathrm{Aut}(\mathfrak{jan}^*) \\ & a & \longmapsto & \mathrm{Ad}^*_a : \mu \longmapsto \left(Z \longmapsto \mu\left(a^{-1}Za\right)\right) \end{array}$$

Proposition 4.1. Let:

$$a := \begin{pmatrix} L & 0 & D \\ 0 & (-1)^\eta & \phi \\ 0 & 0 & 1 \end{pmatrix} \in \mathcal{J}an \ \ ,\ \ \left\{ \ M \ \middle|\ P \ \middle|\ q \ \right\} \in \mathfrak{jan}^*.$$

We have:

$$\begin{aligned} & a \bullet \left\{ \ M \ \middle|\ P \ \middle|\ q \ \right\} \\ &= \left\{ \ LM\tau(L) + D\tau(P)\tau(L) - LP\tau(D) \ \middle|\ LP \ \middle|\ (-1)^\eta q \ \right\}. \end{aligned}$$

[4] For all $\beta \in \mathbb{R}^*$, the application Φ_β which to $M \in \mathcal{A}(1,3)$ associates the linear form $\Lambda \longmapsto \beta\mathrm{Tr}(M\Lambda)$ is an isomorphism of $\mathcal{A}(1,3)$ to $\mathcal{A}(1,3)^*$. Taking $\{A_{kl} := -E_{kl} + [I_{1,3}]_{ll}[I_{1,3}]_{kk}E_{lk},\ k,l \in \{1,\dots,4\},\ k<l\}$ the canonical basis of $\mathcal{A}(1,3)$, we have $\Phi_{-1/2}(A_{kl})(A_{kl}) = 1$, hence the choice of $\beta := -1/2$.

[5] The elements of $\mathfrak{jan}^*$ are called **torsors**.

Proof. We have:

$$
\begin{aligned}
&\left(a \bullet \left\{\ M \ \middle|\ P \ \middle|\ q\ \right\}\right) \begin{pmatrix} \Lambda & 0 & \Gamma \\ 0 & 0 & \varepsilon \\ 0 & 0 & 0 \end{pmatrix} \\
&= \left\{\ M \ \middle|\ P \ \middle|\ q\ \right\} \left(a^{-1} \begin{pmatrix} \Lambda & 0 & \Gamma \\ 0 & 0 & \varepsilon \\ 0 & 0 & 0 \end{pmatrix} a \right) \\
&= \left\{\ M \ \middle|\ P \ \middle|\ q\ \right\} \left(\begin{pmatrix} \tau(L) & 0 & -\tau(L)D \\ 0 & (-1)^\eta & (-1)^{\eta+1}\phi \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \Lambda & 0 & \Gamma \\ 0 & 0 & \varepsilon \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} L & 0 & D \\ 0 & (-1)^\eta & \phi \\ 0 & 0 & 1 \end{pmatrix} \right) \\
&= \left\{\ M \ \middle|\ P \ \middle|\ q\ \right\} \begin{pmatrix} \tau(L)\Lambda L & 0 & \tau(L)(\Lambda D + \Gamma) \\ 0 & 0 & (-1)^\eta \varepsilon \\ 0 & 0 & 0 \end{pmatrix} \\
&= -\frac{1}{2} \mathrm{Tr}\left(M\tau(L)\Lambda L\right) - \tau(P)\tau(L)(\Lambda D + \Gamma) - (-1)^\eta q \varepsilon \\
&= -\frac{1}{2} \mathrm{Tr}\left[\left(LM\tau(L) + 2D\tau(P)\tau(L)\right)\Lambda\right] - \tau(LP)\Gamma - (-1)^\eta q \varepsilon \\
&= -\frac{1}{2} \mathrm{Tr}\left[\left(LM\tau(L) + D\tau(P)\tau(L) - LP\tau(D)\right)\Lambda\right] - \tau(LP)\Gamma - (-1)^\eta q \varepsilon \\
&= \left\{\ LM\tau(L) + D\tau(P)\tau(L) - LP\tau(D) \ \middle|\ LP \ \middle|\ (-1)^\eta q\ \right\} \begin{pmatrix} \Lambda & 0 & \Gamma \\ 0 & 0 & \varepsilon \\ 0 & 0 & 0 \end{pmatrix}
\end{aligned}
$$

□

To describe the Lie algebra of $\mathcal{J}an$, we can also use the isomorphism of Lie algebras[6]:

$$
\begin{array}{cccc}
j: & (\mathbb{R}^3, \wedge) & \longrightarrow & (\mathcal{A}(3), [\ ,\ ]) \\
& \begin{pmatrix} x \\ y \\ z \end{pmatrix} & \longmapsto & \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix}
\end{array} .
$$

with $\wedge$ the cross product on $\mathbb{R}^3$ and $\mathcal{A}(3)$ the vector space of antisymmetric matrices of size 3. Then, we have:

$$
\mathfrak{jan} = \left\{ \begin{pmatrix} \Lambda & 0 & \Gamma \\ 0 & 0 & \varepsilon \\ 0 & 0 & 0 \end{pmatrix},\ \Lambda \in \mathcal{A}(1,3)\ \wedge\ \ \Gamma \in \mathbb{R}^4\ \wedge\ \varepsilon \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} 0 & \beta^T & 0 & v \\ \beta & j(w) & 0 & \gamma \\ 0 & 0 & 0 & \varepsilon \\ 0 & 0 & 0 & 0 \end{pmatrix},\ \beta, w, \gamma \in \mathbb{R}^3\ \wedge\ v, \varepsilon \in \mathbb{R} \right\}.
$$

[6]We have for all $u, v \in \mathbb{R}^3$: $u \wedge v = j(u)(v)$ and $j(u \wedge v) = [j(u), j(v)] = j(u)j(v) - j(v)j(u)$.

Therefore, for all $\left\{\ M \mid P \mid q\ \right\} \in \mathfrak{jan}^*$, there are unique $\ell, g, p \in \mathbb{R}^3$ and $E, q \in \mathbb{R}$ such as:

$$\left\{\ M \mid P \mid q\ \right\} \begin{pmatrix} \Lambda & 0 & \Gamma \\ 0 & 0 & \varepsilon \\ 0 & 0 & 0 \end{pmatrix} = \left\{\ \begin{pmatrix} 0 & g^T \\ g & j(\ell) \end{pmatrix} \middle| \begin{pmatrix} E \\ p \end{pmatrix} \middle| q\ \right\} \begin{pmatrix} 0 & \beta^T & 0 & v \\ \beta & j(w) & 0 & \gamma \\ 0 & 0 & 0 & \varepsilon \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= -\frac{1}{2}\mathrm{Tr}\left(\begin{pmatrix} 0 & g^T \\ g & j(\ell) \end{pmatrix}\begin{pmatrix} 0 & \beta^T \\ \beta & j(w) \end{pmatrix}\right) - \begin{pmatrix} E & p^T \end{pmatrix} I_{1,3} \begin{pmatrix} v \\ \gamma \end{pmatrix} - q\varepsilon$$

$$= \ell^T w - g^T \beta + p^T \gamma - Ev - q\varepsilon$$

We denote this last equality as:

$$\left\{\ \ell \mid g \mid p \mid E \mid q\ \right\} \begin{pmatrix} 0 & \beta^T & 0 & v \\ \beta & j(w) & 0 & \gamma \\ 0 & 0 & 0 & \varepsilon \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The dual $\mathfrak{jan}^*$ has the following descriptions:

$$\left\{ \left\{\ \ell \mid g \mid p \mid E \mid q\ \right\} : \begin{pmatrix} 0 & \beta^T & 0 & v \\ \beta & j(w) & 0 & \gamma \\ 0 & 0 & 0 & \varepsilon \\ 0 & 0 & 0 & 0 \end{pmatrix} \longmapsto \ell^T w - g^T \beta + p^T \gamma - Ev - q\varepsilon,\ \ \ell, g, p \in \mathbb{R}^3 \ \wedge\ E, q \in \mathbb{R} \right\}.$$

Definition 4.3. Let

$$\mu := \left\{\ M \mid P \mid q\ \right\} := \left\{\ l \mid g \mid p \mid E \mid q\ \right\} \in \mathfrak{jan}^*$$

with relations:

$$M = \begin{pmatrix} 0 & g^T \\ g & j(\ell) \end{pmatrix} \quad , \quad P = \begin{pmatrix} E \\ p \end{pmatrix}.$$

(i) The matrix $M := M(\mu) \in \mathcal{A}(1,3)$ is called the *moment matrix associated with* μ. The vector $\ell := \ell(\mu) \in \mathbb{R}^3$ is called the *angular momentum of* M, and the vector $g := g(\mu) \in \mathbb{R}^3$ is the *relativist barycenter of* M.

(ii) (a) The vector $P := P(\mu) \in \mathbb{R}^4$ is called the *stress–energy vector associated with* μ. The vector $p := p(\mu) \in \mathbb{R}^3$ is called the *linear momentum of* P, and the scalar $E := E(\mu) \in \mathbb{R}$ is called the *energy of* P.

(b) The *first Casimir number associated with* μ is defined by:

$$C_1 := C_1(\mu) := P^T I_{1,3} P = E^2 - p^2.$$

(c) The *mass associated to* μ is defined by :

$$m := m(\mu) := \mathrm{sign}(E)\sqrt{C_1} = \mathrm{sign}(E)\sqrt{E^2 - p^2}.$$

(iii) The scalar $q := q(\mu) \in \mathbb{R}$ is called the *electric charge associated with* μ.

We deduce a simple expression of the action of the **CPT**-group $\mathcal{K}$ on the torsors of $\mathfrak{jan}^*$.

Corollary 4.2. Let $\left\{ \; l \;\middle|\; g \;\middle|\; p \;\middle|\; E \;\middle|\; q \; \right\} \in \mathfrak{jan}^*$. For all $\lambda, \eta, \nu \in \{0, 1\}$, we have:

$$(\mathbf{C}^\eta \mathbf{P}^\nu \mathbf{T}^\lambda) \bullet \left\{ \; l \;\middle|\; g \;\middle|\; p \;\middle|\; E \;\middle|\; q \; \right\} = \left\{ \; l \;\middle|\; (-1)^{\lambda+\nu} g \;\middle|\; (-1)^\nu p \;\middle|\; (-1)^\lambda E \;\middle|\; (-1)^\eta q \; \right\}.$$

Proof. We apply the Proposition 4.1 with $a := \mathbf{C}^\eta \mathbf{P}^\nu \mathbf{T}^\lambda$:

$$\begin{aligned}
(\mathbf{C}^\eta \mathbf{P}^\nu \mathbf{T}^\lambda) \bullet \left\{ \; l \;\middle|\; g \;\middle|\; p \;\middle|\; E \;\middle|\; q \; \right\} &= (\mathbf{C}^\eta \mathbf{P}^\nu \mathbf{T}^\lambda) \bullet \left\{ \; \begin{pmatrix} 0 & g^T \\ g & j(\ell) \end{pmatrix} \;\middle|\; \begin{pmatrix} E \\ p \end{pmatrix} \;\middle|\; q \; \right\} \\
&= \left\{ \; \tilde{\mathbf{P}}^\nu \tilde{\mathbf{T}}^\lambda \begin{pmatrix} 0 & g^T \\ g & j(\ell) \end{pmatrix} \tilde{\mathbf{T}}^\lambda \tilde{\mathbf{P}}^\nu \;\middle|\; I_{1,3} \tilde{\mathbf{T}}^\lambda \tilde{\mathbf{P}}^\nu I_{1,3} \begin{pmatrix} E \\ p \end{pmatrix} \;\middle|\; (-1)^\eta q \; \right\} \\
&= \left\{ \; \begin{pmatrix} 0 & (-1)^{\lambda+\nu} g^T \\ (-1)^{\lambda+\nu} g & j(\ell) \end{pmatrix} \;\middle|\; \begin{pmatrix} (-1)^\lambda E \\ (-1)^\nu p \end{pmatrix} \;\middle|\; (-1)^\eta q \; \right\} \\
&= \left\{ \; l \;\middle|\; (-1)^{\lambda+\nu} g \;\middle|\; (-1)^\nu p \;\middle|\; (-1)^\lambda E \;\middle|\; (-1)^\eta q \; \right\}
\end{aligned}$$

□

So we have:

$$\begin{aligned}
\mathbf{C} \bullet \left\{ \; l \;\middle|\; g \;\middle|\; p \;\middle|\; E \;\middle|\; q \; \right\} &= \left\{ \; l \;\middle|\; g \;\middle|\; p \;\middle|\; E \;\middle|\; -q \; \right\} \\
\mathbf{P} \bullet \left\{ \; l \;\middle|\; g \;\middle|\; p \;\middle|\; E \;\middle|\; q \; \right\} &= \left\{ \; l \;\middle|\; -g \;\middle|\; -p \;\middle|\; E \;\middle|\; q \; \right\} \\
\mathbf{T} \bullet \left\{ \; l \;\middle|\; g \;\middle|\; p \;\middle|\; E \;\middle|\; q \; \right\} &= \left\{ \; l \;\middle|\; -g \;\middle|\; p \;\middle|\; -E \;\middle|\; q \; \right\}
\end{aligned}$$

Corollary 4.3. Let $\mu \in \mathfrak{jan}^*$. For all $\lambda, \eta, \nu \in \{0, 1\}$, we have:

$$\begin{aligned}
P\left((\mathbf{C}^\eta \mathbf{P}^\nu \mathbf{T}^\lambda) \bullet \mu\right) &= \tilde{\mathbf{P}}^\nu \tilde{\mathbf{T}}^\lambda P(\mu) \\
C_1\left((\mathbf{C}^\eta \mathbf{P}^\nu \mathbf{T}^\lambda) \bullet \mu\right) &= C_1(\mu) \\
m\left((\mathbf{C}^\eta \mathbf{P}^\nu \mathbf{T}^\lambda) \bullet \mu\right) &= (-1)^\lambda m(\mu)
\end{aligned}$$

Proof. Let $\mu := \left\{ \; l \;\middle|\; g \;\middle|\; p \;\middle|\; E \;\middle|\; q \; \right\} \in \mathfrak{jan}^*$. We have for the stress-energy tensor:

$$\begin{aligned}
P(\mathbf{P} \bullet \mu) &= P\left(\left\{ \; l \;\middle|\; -g \;\middle|\; -p \;\middle|\; E \;\middle|\; q \; \right\}\right) = \begin{pmatrix} E \\ -p \end{pmatrix} = \tilde{\mathbf{P}} P(\mu) \\
P(\mathbf{T} \bullet \mu) &= P\left(\left\{ \; l \;\middle|\; -g \;\middle|\; p \;\middle|\; -E \;\middle|\; q \; \right\}\right) = \begin{pmatrix} -E \\ p \end{pmatrix} = \tilde{\mathbf{T}} P(\mu) \\
P(\mathbf{P} \bullet \mu) &= P\left(\left\{ \; l \;\middle|\; g \;\middle|\; p \;\middle|\; E \;\middle|\; -q \; \right\}\right) = \begin{pmatrix} E \\ p \end{pmatrix} = P(\mu)
\end{aligned}$$

for the first Casimir number:

$$C_1\left((\mathbf{C}^\eta \mathbf{P}^\nu \mathbf{T}^\lambda) \bullet \mu\right) = P(\mu)^T \tilde{\mathbf{T}}^\lambda \tilde{\mathbf{P}}^\nu I_{1,3} \tilde{\mathbf{P}}^\nu \tilde{\mathbf{P}}^\lambda P(\mu) = P(\mu)^T I_{1,3} P(\mu) = C_1(\mu)$$

for the mass:

$$m\left((\mathbf{C}^\eta\mathbf{P}^\nu\mathbf{T}^\lambda)\bullet\mu\right) = \mathrm{sign}\left(E\left((\mathbf{C}^\eta\mathbf{P}^\nu\mathbf{T}^\lambda)\bullet\mu\right)\right)\sqrt{C_1\left((\mathbf{C}^\eta\mathbf{P}^\nu\mathbf{T}^\lambda)\bullet\mu\right)} = \mathrm{sign}((-1)^\lambda E)\sqrt{C_1(\mu)} = (-1)^\lambda m(\mu)$$

$\square$

Therefore the elements variable by these actions are:

$$P\left(\mathbf{P}\bullet\mu\right) = \tilde{\mathbf{P}}P(\mu) \qquad P\left(\mathbf{T}\bullet\mu\right) = \tilde{\mathbf{T}}P(\mu) \qquad m\left(\mathbf{T}\bullet\mu\right) = -m(\mu) \tag{33}$$

and we have the following table:

		$\lambda = 0$	$\lambda = 1$
$\eta = 0$	$\nu = 0$	• neutral symetry • $a = I_6$ • $\mu' = \{\ell, g, p, E, q\}$	• $\mathbf{T}$ – symetry • $a = \mathbf{T}$ • $\mu' = \{\ell, -g, p, -E, q\}$
	$\nu = 1$	• $\mathbf{P}$ – symetry • $a = \mathbf{P}$ • $\mu' = \{\ell, -g, -p, E, q\}$	• $\mathbf{PT}$ – symetry • $a = \mathbf{PT}$ • $\mu' = \{\ell, g, -p, -E, q\}$
$\eta = 1$	$\nu = 0$	• $\mathbf{C}$ – symetry • $a = \mathbf{C}$ • $\mu' = \{\ell, g, p, E, -q\}$	• $\mathbf{CT}$ – symetry • $a = \mathbf{CT}$ • $\mu' = \{\ell, -g, p, -E, -q\}$
	$\nu = 1$	• $\mathbf{CP}$ – symetry • $a = \mathbf{CP}$ • $\mu' = \{\ell, -g, -p, E, -q\}$	• $\mathbf{CPT}$ – symetry • $a = \mathbf{CPT}$ • $\mu' = \{\ell, g, -p, -E, -q\}$

Figure 6: This table lists the 8 values of $\mu' := (\mathbf{C}^\eta\mathbf{P}^\nu\mathbf{T}^\lambda)\bullet\left\{\ l\ \middle|\ g\ \middle|\ p\ \middle|\ E\ \middle|\ q\ \right\}$ for all $\lambda, \eta, \nu \in \{0, 1\}$.

5 Discussion & Conclusion

General relativity immediately and irrevocably rejects the introduction of negative masses into the universe by invoking the resulting Runaway phenomenon, and violations of the principles of action-reaction and equivalence ([5]). Therefore, the construction of a new model introducing states of negative energy and mass could only be considered in relativity by extending its initial geometric context. The theory of symplectic groups, articulated around the different Lorentz, Poincaré or Kaluza groups allows to describe a universe where no force is exerted, a flat world without curvature and where particles follow geodesics of the Minkowski space according to a Lorentzian metric or of a fibered space structured by a fifth dimension (open or closed). This theory only suggests the existence of two kinds of contents that thus live in isolation without interacting.

The transition from the Lorentzian metric to a Riemannian metric, a solution to a field equation arising from an action, introduces these interactions between masses by giving them the power to contribute to the gravitational field. All this translates into the construction of a field equation, the Einstein equation, whose solutions, in the form of metrics, describe the behavior of this universe, either globally, over time, or locally by introducing phenomena of gravitational lenses and the advance of the perihelia of elliptical type trajectories. These phenomena have been confirmed by observation. However, if we have to manage two types of masses, they will have to travel according to their own networks of geodesics structured by distinct metrics in an extended geometric context. This bimetric configuration will then emerge from a couple of two linked field equations that will be presented and studied in a second paper, as well as the related geometric aspects. The phenomena accounted for by this new model will be listed, including in the forefront the acceleration of cosmic expansion. A balance will be made of the agreements of this model with already observed phenomena, as well as predicted phenomena, then observed, such as the early birth of galaxies recently observed by the James Webb Space Telescope. Thus, mathematics would play the role of a guide in relation to physics, illuminating the path to follow, rather than the path already taken.

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